

# Characterizations of the Existence of Partial and Total One-Way Permutations<sup>1</sup>

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## Abstract

In this note, we study the easy certificate classes introduced by Hemaspaandra, Rothe, and Wechsung [HRW], with regard to the question of whether or not surjective one-way functions exist. This is an important open question in cryptology. We show that the existence of partial one-way permutations can be characterized by separating P from the class of UP sets that, for all unambiguous polynomial-time Turing machines accepting them, always have easy (i.e., polynomial-time computable) certificates. This extends results of Grollmann and Selman [GS88]. By Grädel’s recent results about one-way functions [Grä94], this also links statements about easy certificates of NP sets with statements in finite model theory. Similarly, there exist surjective poly-one one-way functions if and only if there is a set  $L$  in P such that not all FewP machines accepting  $L$  always have easy certificates. We also establish a condition necessary and sufficient for the existence of (total) one-way permutations.

## 1 Introduction

What makes NP-complete problems intractable? One possible source of their potential intractability is the fact that there are many possible sets of solutions: The search space is exponential so the cardinality of the set of sets of solutions is double-exponential in the input size. Another possible source of NP’s complexity is that all solutions (even if there are just a few of them) may be random in the sense of Kolmogorov complexity and thus hard to find. For both reasons one may try to “remove” the difficulty from NP by considering subclasses of NP that, by definition, contain only easy sets with respect to either type of difficulty. NP’s subclasses UP (unambiguous polynomial time) [Val76] and FewP (ambiguity-bounded polynomial time) [All86, AR88] both implicitly reduce the richness of the class of potential solutions to  $2^{n^{O(1)}}$ . To single out those NP sets that, for all NP machines accepting them, have easy solutions—i.e., solutions of small Kolmogorov complexity—for all instances in the set, Hemaspaandra, Rothe, and Wechsung [HRW] defined the class  $\text{EASY}_{\forall}^{\forall}$  (see the next section for precise definitions). Interestingly, both these concepts of easy NP sets (to wit, UP and  $\text{EASY}_{\forall}^{\forall}$ ) have their own connection to the invertibility of certain types of one-way functions, as will be stated below. Intuitively, a one-way function is a function that is easy to compute but hard to invert. One-way functions play a central role in complexity-theoretic cryptography [GS88], where the open question of whether such functions do or do not exist is of central importance.

It is well-known that many-one one-way functions exist if and only if  $P \neq NP$ . Thus, we cannot hope for an ultimate solution to the question of whether or not one-way functions exist unless we can solve the famous  $P \stackrel{?}{=} NP$  question. All we can hope for is to characterize the existence of certain special types of one-way functions via complexity-theoretic statements such as the collapse or separation of the corresponding complexity classes. Many types of one-way functions have been studied in the literature. Most notable among such results is Grollmann and Selman’s characterization of the existence of certain types of *injective* one-way functions by conditions such as  $P \neq UP$  or  $P \neq UP \cap \text{coUP}$  [GS88] (see also [Ko85]). Allender extended their results by proving that *poly-one* one-way functions exist if and only if  $P \neq \text{FewP}$  [All86]. Watanabe showed that *constant-one*

one-way functions exist if and only if injective one-way functions exist [Wat88], notwithstanding the fact that even at the level of constant injectivity it has been shown [HH94] that greater injectivity yields strictly more general reductions. Watanabe also showed that the existence of *randomized injective* one-way functions and the existence of *extensible injective* one-way functions, respectively, can be characterized by the separations  $\text{BPP} \neq \text{UP}^{\text{BPP}}$  and  $\mathcal{P} \neq \mathcal{UP}$  [Wat92], where BPP denotes bounded probabilistic polynomial time [Gil77],  $\mathcal{P}$  is the class of polynomial-time solvable promise problems (in the sense of [EY80, ESY84, GS88], see also [HR97]), and  $\mathcal{UP}$  is the class of unambiguous promise problems. Finally, Fenner et al. [FFNR96] proved the existence of *surjective many-one* one-way functions equivalent to  $\mathcal{P} \not\subseteq \text{EASY}_{\forall}^{\forall}$ .

In this note, a characterization of the existence of *injective and surjective* one-way functions is given by separating  $\mathcal{P}$  from a class, denoted  $\text{EASY}_{\forall}^{\forall}(\text{UP})$ , which combines the restriction of unambiguous computation with the constraint required by  $\text{EASY}_{\forall}^{\forall}$ . Thus,  $\text{EASY}_{\forall}^{\forall}(\text{UP})$  *simultaneously* reduces the solution space of NP problems to at most one solution and requires that this one solution can be found and printed out in polynomial time, if it exists. Furthermore, the existence of *surjective poly-one* one-way functions is shown to be equivalent to the separation of  $\mathcal{P}$  and  $\text{EASY}_{\forall}^{\forall}(\text{FewP})$  (which is the polynomially ambiguity-bounded analog of  $\text{EASY}_{\forall}^{\forall}(\text{UP})$ ). Our work is connected to the seemingly unrelated field of (finite model) logic; from Grädel's [Grä94] recent results about one-way functions, we obtain as a corollary equivalences between statements about easy certificates of NP sets and statements in finite model theory such as that the weak definability principle in a logic on finite structures fails to hold. In addition, based on Selman's analogous observation for  $\text{EASY}_{\forall}^{\forall}$  (as cited in [HRW]), further characterizations of the existence of surjective one-way functions are provided in terms of the question of whether  $\text{EASY}_{\forall}^{\forall}(\text{UP})$  and  $\text{EASY}_{\forall}^{\forall}(\text{FewP})$  are closed under complementation. Finally, we show that the existence of total injective one-way functions with a  $\mathcal{P}$ -rankable range is a condition necessary and sufficient for the existence of one-way permutations.

## 2 Preliminaries

All sets considered are subsets of  $\Sigma^*$ , where  $\Sigma = \{0, 1\}$ . Functions map from  $\Sigma^*$  to  $\Sigma^*$  and are many-one and partial (unless explicitly specified to be one-one or total). The length of a string  $x \in \Sigma^*$  is denoted by  $|x|$  and the cardinality of a set  $L \subseteq \Sigma^*$  by  $\|L\|$ . Let  $\epsilon$  denote the empty string. Let  $\langle \cdot, \cdot \rangle$  be a standard easily computable pairing function (i.e., a bijection between  $\Sigma^* \times \Sigma^*$  and  $\Sigma^*$ ) that can be extended to encode tuples of strings by one string as usual. Let  $\leq_{\text{lex}}$  denote the standard quasi-lexicographical ordering on  $\Sigma^*$ . For each (single-valued, partial or total) function  $f : \Sigma^* \rightarrow \Sigma^*$ , let  $\text{dom}(f)$  and  $\text{range}(f)$  denote the domain and range of  $f$ , respectively.

Let NPM be a shorthand for “nondeterministic polynomial-time Turing machine.” For each NPM  $M$ ,  $L(M)$  denotes the language accepted by  $M$ . For each NPM  $M$  and any input  $x$ , we denote the set of accepting paths of  $M(x)$  by  $\text{acc}_M(x)$ . An NPM  $M$  is a UP machine (FewP machine, respectively) if, for all inputs  $x$ ,  $M(x)$  has at most one (at most polynomially in  $|x|$ , respectively) accepting paths, and  $M$  accepts  $x$  if and only if  $M(x)$  has at least one accepting path. UP [Val76] (respectively, FewP [All86, AR88]) is the class of sets  $L$  such that  $L = L(M)$  for some UP machine (FewP machine)  $M$ . FP denotes the class of polynomial-time computable functions.  $\text{EASY}_{\forall}^{\forall}$  [HRW] is defined to be the class of all sets  $L$  for which all NPMs accepting  $L$  always (i.e.,

on all inputs  $x \in L$ ) have easy certificates (i.e., accepting paths whose encoding can be printed in polynomial time). Of the four classes  $\text{EASY}_{\forall}^{\forall}$ ,  $\text{EASY}_{\text{io}}^{\forall}$ ,  $\text{EASY}_{\forall}^{\exists}$ , and  $\text{EASY}_{\text{io}}^{\exists}$  considered by Hemaspaandra, Rothe, and Wechsung [HRW], only  $\text{EASY}_{\forall}^{\forall}$  is relevant for the characterization of one-way functions.

Now let us formally define the UP and FewP analog of  $\text{EASY}_{\forall}^{\forall}$ . Though it is clear that a more general definition of the form  $\text{EASY}_{\forall}^{\forall}(\mathcal{C}, \mathcal{F})$  for complexity classes  $\mathcal{C}$  other than NP, UP, or FewP and for function classes  $\mathcal{F}$  other than FP can analogously be obtained, we will only define the classes of interest here.

**Definition 2.1** For  $\mathcal{C} \in \{\text{NP}, \text{UP}, \text{FewP}\}$ , define  $\text{EASY}_{\forall}^{\forall}(\mathcal{C})$  to be the class of all sets  $L$  that either are finite, or that satisfy (a)  $L \in \mathcal{C}$ , and (b) for every  $\mathcal{C}$ -machine  $N$  such that  $L(N) = L$ , there exists an FP function  $f_N$  such that, for all  $x \in L$ ,  $f_N(x) \in \text{acc}_N(x)$ .

The inclusions summarized in Proposition 2.2 below follow immediately from the definition. For instance, the inclusion  $\text{EASY}_{\forall}^{\forall} \subseteq \text{EASY}_{\forall}^{\forall}(\text{FewP})$  holds, since each  $\text{EASY}_{\forall}^{\forall}$  set  $L$  is in P (see [HRW, Figure 1]) and thus in FewP, and moreover since if every NPM accepting  $L$  always has easy certificates, then so does every FewP machine. The inclusion  $\text{EASY}_{\forall}^{\forall}(\text{UP}) \subseteq \text{P}$  holds due to  $\text{EASY}_{\forall}^{\forall}(\text{UP}) \subseteq \text{EASY}_{\forall}^{\exists}(\text{UP}) = \text{EASY}_{\forall}^{\exists} = \text{P}$  (see [HRW, Theorem 2.2.1]), where  $\text{EASY}_{\forall}^{\exists}(\text{UP})$  denotes the analog of  $\text{EASY}_{\forall}^{\forall}(\text{UP})$  such that condition (b) in Definition 2.1 above is required to hold only for *some* UP machine  $N$ .

**Proposition 2.2**  $\text{EASY}_{\forall}^{\forall} \subseteq \text{EASY}_{\forall}^{\forall}(\text{FewP}) \subseteq \text{EASY}_{\forall}^{\forall}(\text{UP}) \subseteq \text{P} \subseteq \text{UP} \subseteq \text{FewP} \subseteq \text{NP}$ .

Next, we define the types of one-way functions considered in this paper. Note that the honesty of one-way functions is required in order to avoid the case that the FP-noninvertibility is trivial.

- Definition 2.3**
1. A function  $f$  is *honest* if there is a polynomial  $p$  such that for every  $y \in \text{range}(f)$  and for every  $x \in \text{dom}(f)$ , if  $y = f(x)$  then  $|x| \leq p(|y|)$ .
  2. A function  $f$  is *poly-one* if there is a polynomial  $p$  such that  $\|f^{-1}(y)\| \leq p(|y|)$  for each  $y \in \text{range}(f)$ .
  3. A (many-one) function  $f$  is said to be *FP-invertible* if there is a function  $g \in \text{FP}$  such that for every  $y \in \text{range}(f)$ ,  $g(y)$  prints *some* value of  $f^{-1}(y)$ . In particular, if  $f$  is one-one, FP-invertibility of  $f$  means  $f^{-1} \in \text{FP}$ .
  4. A function  $f$  is said to be a *one-one* (respectively, *poly-one*, *many-one*) *one-way function* if  $f$  is honest, one-one (respectively, poly-one, many-one),  $f \in \text{FP}$ , and  $f$  is not FP-invertible. If  $f : \Sigma^* \rightarrow \Sigma^*$  is a total, surjective, and one-one one-way function,  $f$  is called a *one-way permutation*.

Sometimes the following weaker definition of honesty is used:  $f$  is *honest* if there is a polynomial  $p$  such that for every  $y \in \text{range}(f)$  there is a string  $x \in \text{dom}(f)$  such that  $y = f(x)$  and  $|x| \leq p(|y|)$ . All claims in this paper, except those involving weak one-way functions (defined later), hold also for this alternate definition.

Note that we discuss one-way functions in the complexity-theoretic setting introduced by Grollmann and Selman [GS88]. So-called cryptographic one-way functions are not discussed here, though we should mention that one-way permutations have been interestingly studied in that context [Yao82, IR89, HILL91].

### 3 Characterizing the Existence of Surjective One-Way Functions

Fenner et al. [FFNR96] have characterized the existence of surjective *many-one* one-way functions by the condition  $P \not\subseteq \text{EASY}_{\forall}^{\forall}$ . In this section, we give analogous characterizations of the existence of surjective one-one one-way functions and surjective poly-one one-way functions by separating P from  $\text{EASY}_{\forall}^{\forall}(\text{UP})$  and  $\text{EASY}_{\forall}^{\forall}(\text{FewP})$ , respectively. We mention here that there is a relativization in which  $P \not\subseteq \text{EASY}_{\forall}^{\forall}$  does not imply  $P \neq \text{NP} \cap \text{coNP}$  ([IN88], see also [CS93, FR94, FFNR96]).

We begin with the characterization of the existence of surjective one-one one-way functions. Note that the type of function discussed in item (2) of Theorem 3.1 below is the partial-function analog of a (total) one-way permutation. Note also that the equivalence of statements (1), (3), and (4) in Theorem 3.1 holds in analogy to the case of  $\text{EASY}_{\forall}^{\forall}$  (see [HRW, FFNR96]).

**Theorem 3.1** The following are equivalent.

1.  $\text{EASY}_{\forall}^{\forall}(\text{UP}) \neq P$ .
2. There exists a partial one-one one-way function  $f$  with  $\text{range}(f) = \Sigma^*$ .
3.  $\Sigma^* \notin \text{EASY}_{\forall}^{\forall}(\text{UP})$ .
4.  $\text{EASY}_{\forall}^{\forall}(\text{UP})$  is not closed under complementation.

**Proof.** Clearly, (3) implies (4), since  $\overline{\Sigma^*} = \emptyset$  as a finite set is in  $\text{EASY}_{\forall}^{\forall}(\text{UP})$ . (4) immediately implies (1). To see that (1) implies (3), assume there is a set  $L \in P$  such that  $L \notin \text{EASY}_{\forall}^{\forall}(\text{UP})$ . Let  $N$  be some UP machine accepting  $L$  such that no FP function exists that outputs the accepting path of  $N(x)$  for all inputs  $x \in L$ . Let  $M$  be some P machine that accepts  $\overline{L}$ . Consider the following NPM  $N'$ : On input  $x$ ,  $N'$  guesses whether  $x \in L$  or  $x \in \overline{L}$ . If the guess was “ $x \in L$ ,”  $N'$  simulates  $N(x)$ ; otherwise, it simulates  $M(x)$ . Then,  $N'$  is a UP machine accepting  $\Sigma^*$ . Note that the accepting computation of  $N'(x)$  for inputs  $x \in L$  contains the accepting computation of  $N(x)$ . Since  $L$  cannot be empty (in fact,  $L$  cannot be finite, for otherwise we would have had  $L \in \text{EASY}_{\forall}^{\forall}(\text{UP})$ ), no FP function can output, for all inputs  $x \in \Sigma^*$ , the accepting path of  $N'(x)$ . Hence,  $\Sigma^* \notin \text{EASY}_{\forall}^{\forall}(\text{UP})$ .

(3) implies (2): Assume  $\Sigma^* \notin \text{EASY}_{\forall}^{\forall}(\text{UP})$ . Let  $M$  be a UP machine accepting  $\Sigma^*$  such that no FP function can output the accepting path of  $M(y)$  for all  $y \in \Sigma^*$ . For any input  $y$ , let  $\text{comp}_M(y)$  denote the unique accepting path (encoded as a sequence of configurations) of  $M(y)$ . As in [GS88], define the function  $f$  to be

$$f(x) \stackrel{\text{df}}{=} \begin{cases} y & \text{if } x = \text{comp}_M(y) \\ \perp & \text{otherwise,} \end{cases}$$

where  $\perp$  is a special symbol indicating, in the usage “ $f(x) = \perp$ ,” that  $f$  on  $x$  is not defined. Clearly, given  $x$ , it can be checked in polynomial time whether  $x$  encodes an accepting path of  $M$  (by checking whether it starts with the initial configuration of  $M$  for some input string, all transitions from one configuration to the next are legal, and the final configuration contains an accepting final state), and if so, the input string  $y$  of  $M$  can easily be determined. Thus,  $f \in \text{FP}$ . Since  $M$  is a UP machine,  $f$  is injective. The polynomial bounding the running time of  $M$  witnesses the honesty of  $f$ . Since  $L(M) = \Sigma^*$ ,  $f$  is surjective. Finally,  $f^{-1} \notin \text{FP}$ , since  $f^{-1}(y) = x$  is an accepting computation of  $M(y)$  for each  $y$ , and so  $f^{-1} \in \text{FP}$  contradicts our assumption that  $M$  only has hard certificates. To summarize,  $f$  is a partial one-one one-way function with  $\text{range}(f) = \Sigma^*$ .

(2) implies (3): Let  $f$  be a partial one-one one-way function with  $\text{range}(f) = \Sigma^*$ . We will show that  $\text{range}(f) = \Sigma^*$  is not in  $\text{EASY}_{\forall}^{\forall}(\text{UP})$ . Let  $p$  be the polynomial that witnesses the honesty of  $f$ . Consider the following machine  $M$ . On input  $y$ ,  $M$  nondeterministically guesses all strings  $x$  of length at most  $p(|y|)$ , computes  $f(x)$  for each guessed  $x$ , and accepts  $y$  if and only if  $f(x) = y$ . Clearly,  $M$  is a UP machine accepting  $\Sigma^*$ , since  $f$  is a  $p$ -honest bijection (from some subset of  $\Sigma^*$  onto  $\Sigma^*$ ) computable in polynomial time. Since  $f^{-1} \notin \text{FP}$  and the accepting path of  $M(y)$  contains  $x = f^{-1}(y)$ , no FP function can output, for all  $y$ , the accepting path of  $M$  on input  $y$ . Thus,  $\Sigma^* \notin \text{EASY}_{\forall}^{\forall}(\text{UP})$ .  $\square$

By Grollmann and Selman’s characterization of the existence of partial one-one one-way functions with  $\text{range}(f) = \Sigma^*$  [GS88], we immediately have Corollary 3.2, which has previously been proven directly by Hartmanis and Hemaspaandra (then Hemachandra) [HH88], using different notation. As a point of interest, we note that Corollary 3.2 proves that separating P from a certain class containing P is equivalent to separating P from a certain class contained in P. Also, though Naor and Impagliazzo [IN88, Proposition 4.2] (see also [CS93, FR94, FFNR96]) have shown that for the converse of the original (i.e., NP) version of the Borodin-Demers [BD76] theorem<sup>1</sup> there is a relativized counterexample, Corollary 3.2 says that the converse of the UP analog of the Borodin-Demers theorem holds (see [HH88] for discussion of this point).

**Corollary 3.2** [HH88]  $\text{P} \neq \text{UP} \cap \text{coUP}$  if and only if  $\text{EASY}_{\forall}^{\forall}(\text{UP}) \neq \text{P}$ .

A seemingly unrelated connection comes from finite model theory. Grädel [Grä94] has recently shown that  $\text{P} = \text{UP} \cap \text{coUP}$  if and only if the weak definability principle holds for every first order logic  $\mathcal{L}$  on finite structures that captures P. The weak definability principle says: Every totally defined query (on the set of finite structures of the relations of a first order logic  $\mathcal{L}$ ) that is implicitly definable in  $\mathcal{L}$  is also explicitly definable in  $\mathcal{L}$  (see [Grä94] for those notions not defined here).

**Corollary 3.3**  $\text{EASY}_{\forall}^{\forall}(\text{UP}) \neq \text{P}$  if and only if the weak definability principle fails for some first order logic  $\mathcal{L}$  on finite structures that captures P.

Fenner et al. [FFNR96] also consider the “one-bit version” of the condition  $\Sigma^* \in \text{EASY}_{\forall}^{\forall}$ . Let us define  $1\text{-EASY}_{\forall}^{\forall}(\mathcal{C})$  to be the class of all sets  $L$  that either are finite, or that satisfy (a)  $L \in \mathcal{C}$ , and (b) for every  $\mathcal{C}$ -machine  $N$  such that  $L(N) = L$ , there exists an FP function  $f_N$  such that, for all

<sup>1</sup>Which says  $\text{P} \neq \text{NP} \cap \text{coNP}$  implies  $\text{EASY}_{\forall}^{\forall} \neq \text{P}$ , except it states this in a different but equivalent form.

$x \in L$ ,  $f_N(x)$  outputs the first bit of an (“the” in the case  $\mathcal{C} = \text{UP}$ ) accepting path of  $N(x)$ . Clearly (as in the case of NP), we have for the UP case: (a)  $P = \text{EASY}_{\forall}^{\forall}(\text{UP})$  implies  $P = 1\text{-EASY}_{\forall}^{\forall}(\text{UP})$ , and (b)  $P = 1\text{-EASY}_{\forall}^{\forall}(\text{UP})$  implies  $P = \text{UP} \cap \text{coUP}$ . Thus, Corollary 3.2 in fact can be restated as Corollary 3.4, which sharply contrasts with the NP case [IN88, FR94, FFNR96], i.e., even though  $P = \text{EASY}_{\forall}^{\forall}$ ,  $P = 1\text{-EASY}_{\forall}^{\forall}$ , and  $P = \text{NP} \cap \text{coNP}$  appear to be pairwise different conditions, their UP variants behave equivalently, and thus it is not reasonable to consider a “one-bit version” of  $\text{EASY}_{\forall}^{\forall}(\text{UP})$ .

**Corollary 3.4** (see also [HH88]) The collapses  $P = \text{EASY}_{\forall}^{\forall}(\text{UP})$ ,  $P = 1\text{-EASY}_{\forall}^{\forall}(\text{UP})$ , and  $P = \text{UP} \cap \text{coUP}$  are pairwise equivalent.

Now we characterize the existence of surjective poly-one one-way functions by separating P and  $\text{EASY}_{\forall}^{\forall}(\text{FewP})$ .

**Theorem 3.5** The following are equivalent.

1. There exists a partial surjective poly-one one-way function.
2. There exists a total surjective poly-one one-way function.
3. There exists a total poly-one one-way function  $f$  with  $\text{range}(f) \in P$ .
4. There exists a partial poly-one one-way function  $f$  with  $\text{range}(f) \in P$ .
5.  $\text{EASY}_{\forall}^{\forall}(\text{FewP}) \neq P$ .
6.  $\Sigma^* \notin \text{EASY}_{\forall}^{\forall}(\text{FewP})$ .
7.  $\text{EASY}_{\forall}^{\forall}(\text{FewP})$  is not closed under complementation.

**Proof.** Clearly, (1) implies (3), as if  $f$  is a function satisfying (1), then

$$g(x) \stackrel{\text{df}}{=} \begin{cases} 0f(x) & \text{if } f(x) \neq \perp \\ 1x & \text{if } f(x) = \perp \end{cases}$$

satisfies (3). Also, (3) trivially implies (4).

(4) implies (5): Let  $f$  be a partial poly-one one-way function with  $\text{range}(f) \in P$ . We will show that  $\text{range}(f)$  is not in  $\text{EASY}_{\forall}^{\forall}(\text{FewP})$ . Let  $p$  be the polynomial that witnesses the honesty of  $f$ . Consider the following machine  $M$ . On input  $y$ ,  $M$  nondeterministically guesses all strings  $x$  of length at most  $p(|y|)$ , computes  $f(x)$  for each guessed  $x$ , and accepts  $y$  if and only if  $f(x) = y$ . Clearly,  $M$  is a FewP machine accepting  $\text{range}(f)$ , since  $f$  is a  $p$ -honest poly-one function computable in polynomial time. Since  $f$  is not FP-invertible and each accepting path of  $M(y)$  contains some value of  $f^{-1}(y)$ , no FP function can output, for all  $y$ , some accepting path of  $M$  on input  $y$ . Thus,  $\text{range}(f) \notin \text{EASY}_{\forall}^{\forall}(\text{FewP})$ .

It is clear that (2) implies (1). Suppose (1) holds, and  $f$  is a function satisfying (1). Then  $f'$  is a function satisfying (2), where

$$f'(x) \stackrel{\text{df}}{=} \begin{cases} \epsilon & \text{if } x = \epsilon \\ f(z)0 & \text{if } x = z0 \text{ and } f(z) \neq \perp \\ z1 & \text{if } x = z0 \text{ and } f(z) = \perp \\ z1 & \text{if } x = z1. \end{cases}$$

The proof that conditions (5), (6), and (7) of this theorem are pairwise equivalent goes through as in the proof of the corresponding claim for  $\text{EASY}_{\forall}^{\forall}$  or  $\text{EASY}_{\forall}^{\forall}(\text{UP})$  (see Theorem 3.1). Finally, that (7) implies (1) can again be seen as in the proof of Theorem 3.1, the only difference being that  $M$  now is a FewP machine accepting  $\Sigma^*$  and the function  $f$  is now defined by  $f(x) = y$  if  $x$  is *some* accepting path of  $M(y)$ , and  $f(x)$  is undefined otherwise. Then,  $f$  is a partial surjective poly-one one-way function. This completes the proof that all statements of the theorem are equivalent.  $\square$

Note that  $P \neq \text{FewP}$  is clearly implied by each of the conditions of Theorem 3.5. Note also that  $P \neq \text{FewP} \cap \text{coFewP}$  clearly implies each of the conditions of Theorem 3.5, though it is not known whether the converse holds. We conjecture that it does not (equivalently, we conjecture that the converse of the FewP analog of the Borodin-Demers theorem does not hold). Thus, the conditions of Theorem 3.5 are intermediate between the conditions  $P \neq \text{FewP} \cap \text{coFewP}$  and  $P \neq \text{FewP}$ .<sup>2</sup>

Could it be the case that the conditions of Theorem 3.5 in fact either are equivalent to  $P \neq \text{FewP} \cap \text{coFewP}$ , or are equivalent to  $P \neq \text{FewP}$ ? Relativized counterexamples are known for each of these cases. In particular, there is a relativized world, constructed by Fortnow and Rogers [FR94], in which the conditions of Theorem 3.5 fail yet  $P \neq \text{FewP}$  holds. Also, Lance Fortnow [For97] has informed us that, using the techniques of Fortnow and Rogers [FR94], one can build a relativized world in which in which  $P = \text{FewP} \cap \text{coFewP}$  yet the conditions of Theorem 3.5 hold.

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<sup>2</sup> Regarding the condition  $P \neq \text{FewP}$ , Allender [All86] showed that the following conditions are all equivalent: (a)  $P \neq \text{FewP}$ , (b) there exists a total poly-one one-way function, and (c) there exists a total poly-one weak one-way function. Weak one-way functions mean the following. A poly-one function  $f$  is *strongly FP-invertible* if there is a function  $g \in \text{FP}$  such that for every  $y \in \text{range}(f)$ ,  $g(y)$  prints *all* elements of  $f^{-1}(y)$ . A function  $f$  is called a *weak one-way function* if  $f \in \text{FP}$ ,  $f$  is poly-one,  $f$  is honest, and  $f$  is not strongly FP-invertible.

Similarly, it is not hard to see, e.g., from Allender's proof, that also equivalent to (a), (b), and (c) are each of these conditions: (d) there exists a total poly-one weak one-way function  $f$  with  $\text{range}(f) \in P$ , and (e) there exists a partial poly-one one-way function.

We note that the following condition is also equivalent to each of (a)–(e): (f) there exists a total surjective poly-one weak one-way function. This is true for the following reasons. Clearly (f) implies (d). Also, (e) implies (f) as if  $h$  is a function satisfying (e), then  $h'$  satisfies (f), where

$$h'(x) \stackrel{\text{df}}{=} \begin{cases} \epsilon & \text{if } x \in \{\epsilon, 0, 1\} \\ h(z)0 & \text{if } x = z00 \text{ and } h(z) \neq \perp \\ z1 & \text{if } (x = z00 \wedge h(z) = \perp) \text{ or } x = z11 \\ z0 & \text{if } x = z01 \text{ or } x = z10. \end{cases}$$



## 4 Characterizing the Existence of One-Way Permutations

For many types of one-way functions, the existence question has been characterized in the literature as equivalent to the separation of suitable complexity classes. Such a characterization for the existence of one-way permutations, however, is still missing. To date, the result closest to this goal is the above-mentioned characterization of the existence of a *partial, injective, and surjective* one-way function  $f$  by the condition  $P \neq UP \cap \text{coUP}$  [GS88].<sup>3</sup> Since  $f$  is not total,  $f$  is not a permutation of  $\Sigma^*$  (even though  $f$  is a bijection mapping a subset of  $\Sigma^*$  onto  $\Sigma^*$ ). Thus,  $P \neq UP \cap \text{coUP}$  potentially is a strictly weaker condition than the existence of a one-way permutation. Of course, such a function  $f$  can be made total [GS88], but only at the cost of loss of surjectivity (even though such a total one-way function created from  $f$  still has a range in  $P$ ). However, we will show below that the existence of one-way permutations is equivalent to the existence of total injective one-way functions whose range is  $P$ -rankable.

**Definition 4.1** [GS91] A set  $A$  is said to be *P-rankable* if there exists a polynomial-time computable function  $\text{rank}$  so that  $(\forall x \in \Sigma^*) [\text{rank}(x) = \|A^{\leq_{\text{lex}} x}\|]$ , where  $A^{\leq_{\text{lex}} x}$  denotes the set of all strings  $w \in A$  with  $w \leq_{\text{lex}} x$ .

That is, a ranking function for  $A$  tells us the number of strings in  $A$  up to a given string. To avoid confusion, we mention that the notion of  $P$ -rankability used here (and in [GS91]) is also sometimes referred to as “strong  $P$ -rankability” (e.g., in [HR90]).

**Theorem 4.2** One-way permutations exist if and only if there exist total one-one one-way functions whose range is  $P$ -rankable.

**Proof.** The “only if” direction is immediate, since  $\Sigma^*$  is  $P$ -rankable.

For the converse, suppose there exists a total one-one one-way function  $f$  whose range is  $P$ -rankable. We will define a one-way permutation  $h$ . Intuitively, the idea is to fill in the holes in the range of  $f$ , using its  $P$ -rankability. Let  $T = \text{range}(f)$  be  $P$ -rankable. For each  $n$ , let  $\text{holes}(n) \stackrel{\text{df}}{=} 2^n - \|T^{\leq n}\|$ . Note that since  $T$  is  $P$ -rankable,  $\text{holes}$  is in  $\text{FP}$ . Let us introduce some useful notation. For each string  $x$ , let  $k(x)$  be the lexicographical position of  $x$  among the length  $|x|$  strings; e.g.,  $k(000) = 1$  and  $k(111) = 8$ . For each string  $x$  and each  $j \in \mathbb{N}$ , let  $x - j$  denote the string that in lexicographical order comes  $j$  places before  $x$ . For each set  $A$  and each  $k \in \mathbb{N}$ , let  $A_{[k]}$  be the  $k$ th string of  $A$  in lexicographical order. Now define the function  $h$  by

$$h(x) \stackrel{\text{df}}{=} \begin{cases} f(x - \sum_{i=0}^{|x|} \text{holes}(i)) & \text{if } k(x) > \text{holes}(|x|) \\ (T \cap \Sigma^{|x|})_{[k(x)]} & \text{if } k(x) \leq \text{holes}(|x|). \end{cases}$$

Since  $T$  is  $P$ -rankable and  $f \in \text{FP}$ , we have  $h \in \text{FP}$ . Clearly,  $h$  is honest and injective,  $h$  is total, and  $\text{range}(h) = \Sigma^*$ . If one could invert  $h$  in polynomial time, then  $f$  would also be  $\text{FP}$ -invertible, as the

<sup>3</sup>Fenner et al. [FFNR96] make the following claim: If  $P = 1\text{-EASY}_{\forall}^{\forall}$ , then there exist no one-way permutations. However, since  $P = 1\text{-EASY}_{\forall}^{\forall}$  implies  $P = UP \cap \text{coUP}$ , the following also correct claim is stronger: If  $P = UP \cap \text{coUP}$ , then there exist no one-way permutations. The difficult part seems to be the converse implication, and we conjecture that the converse does not hold.

<i>Partial functions</i>	one-one	poly-one
no restriction	$P \neq UP$ [GS88]	$P \neq FewP$ (Footnote 2)
surjective	$P \neq EASY_{\forall}^{\forall}(UP)$ (Thm. 3.1)	$P \neq EASY_{\forall}^{\forall}(FewP)$ (Thm. 3.5)
range in P	$P \neq EASY_{\forall}^{\forall}(UP)$ (Thm. 3.1 plus [GS88, Theorem 8])	$P \neq EASY_{\forall}^{\forall}(FewP)$ (Thm. 3.5)

Table 1: Characterizations of the existence of various types of one-way functions: the partial function case.

<i>Total functions</i>	one-one	poly-one
no restriction	$P \neq UP$ [GS88]	$P \neq FewP$ [All86]
surjective	open question (but note Thm. 4.2)	$P \neq EASY_{\forall}^{\forall}(FewP)$ (Thm. 3.5)
range in P	$P \neq EASY_{\forall}^{\forall}(UP)$ (Thm. 3.1 plus [GS88, Theorem 8])	$P \neq EASY_{\forall}^{\forall}(FewP)$ (Thm. 3.5)
weak	$P \neq UP$ [GS88]	$P \neq FewP$ [All86]
surj. & weak	open question (but note Thm. 4.2)	$P \neq FewP$ (Footnote 2)
P-range & weak	$P \neq EASY_{\forall}^{\forall}(UP)$ (Thm. 3.1 plus [GS88, Theorem 8])	$P \neq FewP$ (Footnote 2)

Table 2: Characterizations of the existence of various types of one-way functions: the total function case.

P-rankability of  $T$  allows one to find the string in the range of  $f$  that should be inverted with respect to  $h$ , and after inverting we shift the inverse with respect to  $h$ , say  $z$ , by  $\sum_{i=0}^{|z|} holes(i)$  positions to obtain the true inverse with respect to  $f$ . Hence,  $h$  is a one-way permutation.  $\square$

Note that P-rankability of the range of  $f$  suffices to give us Theorem 4.2, and Theorem 4.2 is stated in this way. However, even weaker notions would work. Without going into precise details, we remark that one just needs a function that, from some easily found and countable set of places, is an honest *address function* (see [GHK92]) for the complement of the range of  $f$ . Of course, the ultimate goal is to find a characterization of the existence of one-way permutations in terms of a separation of suitable complexity classes.

Finally, Tables 1 and 2 summarize the characterization results that are known from the literature and from this paper. Note that for one-one functions, FP-invertibility and strong FP-invertibility are clearly identical notions, and so the one-one column of Tables 1 and 2 is not affected by the “weak” issue.

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